Temporal Logic

Introduction

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Temporal validities

VALIDITIES

THEOREM: The following formulas ($K_{\mathbf{G}}$, $K_{\mathbf{H}}$) are valid on all frames:

$$\forall \mathfrak{F} \qquad \mathfrak{F} \models \mathbf{G}(\varphi \to \psi) \to (\mathbf{G}\varphi \to \mathbf{G}\psi)$$

$$\forall \mathfrak{F} \qquad \mathfrak{F} \models \mathbf{H}(\varphi \to \psi) \to (\mathbf{H}\varphi \to \mathbf{H}\psi)$$

PROOF: Suppose that it is not: then there is a countermodel $\mathfrak{M} = (\mathfrak{F}, v)$ in which for some world w

$$\mathfrak{M}, w \not\models \mathbf{G}(\varphi \to \psi) \to (\mathbf{G}\varphi \to \mathbf{G}\psi)$$

that is,

$$\mathfrak{M}, w \models \mathbf{G}(\varphi \to \psi) \qquad \text{but} \qquad \mathfrak{M}, w \not\models \mathbf{G}\varphi \to \mathbf{G}\psi$$

therefore

$$\underbrace{\mathfrak{M}, w \models \mathbf{G}(\varphi \to \psi)}_{(1)} \qquad \text{and} \qquad \underbrace{\mathfrak{M}, w \models \mathbf{G}\varphi}_{(2)} \qquad \text{but} \qquad \underbrace{\mathfrak{M}, w \not\models \mathbf{G}\psi}_{(3)}$$

(Draw!) Since (3), there is some $v \mathfrak{R} w$ s.t. $\mathfrak{M}, v \not\models \psi$ but according to (2), $\mathfrak{M}, v \models \varphi$. So in v, $\mathfrak{M}, v \not\models \varphi \to \psi$ which contradicts to (1). The proof is the same for $\mathbf{H}$. $\square$

VALIDITIES

THEOREM: The following formulas are valid on all frames for $\Box \in \{\mathbf{G}, \mathbf{H}\}$, $\Diamond \in \{\mathbf{F}, \mathbf{P}\}$:

$$\forall \mathfrak{F} \qquad \mathfrak{F} \models \Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$$

$$\forall \mathfrak{F} \qquad \mathfrak{F} \models \Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi)$$

PROOF: Homework.

VALIDITIES

THEOREM: The following formulas are valid on all frames:

$$\forall \mathfrak{F} \qquad \mathfrak{F} \models \varphi \to \mathbf{GP}\varphi$$

$$\forall \mathfrak{F} \qquad \mathfrak{F} \models \varphi \to \mathbf{HF}\varphi$$

PROOF: Suppose it is not: Then there is a countermodel $\mathfrak{M} = (\mathfrak{F}, v)$ in which for some world w

$$\mathfrak{M}, w \not\models \varphi \to \mathbf{GP}\varphi,$$

i.e.,

$$\mathfrak{M}, w \models \varphi \quad \text{but} \quad \mathfrak{M}, w \not\models \mathbf{GP}\varphi$$

from the latter follows that there is a $v Я w$, $\mathfrak{M}, v \not\models \mathbf{P}\varphi$, i.e., there is no arrow to v from a world in which φ is true. But w is exactly a world like that. The other proof is similar. □

THREE KINDS OF TRUTH

For a formula φ, there are three concept of being true

(a) true in a **world** of a **model**

$$\mathfrak{M}, w \models \varphi$$

(b) true in a **model**

$$\mathfrak{M} \models \varphi \iff \forall w \quad \mathfrak{M}, w \models \varphi$$

(c) valid on a **frame**

$$\mathfrak{F} \models \varphi \iff \forall V \forall w \quad \underbrace{\mathfrak{F}, V}_{\mathfrak{M}}, w \models \varphi$$

$$(a) \Longleftarrow (b) \Longleftarrow (c)$$

THREE KINDS OF TRUTH

For a rule $\dfrac{\begin{array}{c}\varphi_1\\ \vdots\\ \varphi_n\end{array}}{\varphi}$ there are three concept of consequence

(a) **Worldwise** consequence: $\forall\mathfrak{M}\,\forall w\ \dfrac{\begin{array}{ccc}\mathfrak{M},w & \models & \varphi_1\\ \vdots & & \\ \mathfrak{M},w & \models & \varphi_n\end{array}}{\begin{array}{ccc}\mathfrak{M},w & \models & \varphi\end{array}}$

(b) **Modelwise** consequence: $\forall\mathfrak{M}\ \dfrac{\begin{array}{ccc}\mathfrak{M} & \models & \varphi_1\\ \vdots & & \\ \mathfrak{M} & \models & \varphi_n\end{array}}{\begin{array}{ccc}\mathfrak{M} & \models & \varphi\end{array}}$

(c) **Framewise** consequence: $\dfrac{\begin{array}{ccc}\mathfrak{F} & \models & \varphi\\ \vdots & & \\ \mathfrak{F} & \models & \varphi_n\end{array}}{\begin{array}{ccc}\mathfrak{F} & \models & \varphi\end{array}}$

$$(a) \Longrightarrow (b) \Longrightarrow (c)$$

RULES

<u>THEOREM</u>: The following rule (MP), the Modus Ponens is sound **worldwise**:

$$\frac{\begin{array}{l}\mathfrak{M}, w \models \varphi \\ \mathfrak{M}, w \models \varphi \to \psi\end{array}}{\mathfrak{M}, w \models \psi}$$

<u>PROOF</u>: Trivial.

SOUND RULES

THEOREM: The following rule (RN), the Rule of Necessitation is sound **modelwise**:

$$\forall \mathfrak{M} \qquad \mathfrak{M} \models \varphi \implies \mathfrak{M} \models \mathbf{G}\varphi$$

$$\forall \mathfrak{M} \qquad \mathfrak{M} \models \varphi \implies \mathfrak{M} \models \mathbf{H}\varphi$$

PROOF: $\mathfrak{M} \models \varphi$ means that φ is true in all worlds in the model. Then of course it will be true in all worlds that φ will be true in all of its neighbours as well, i.e., $\mathfrak{M}, w \models \mathbf{G}\varphi$ which means the conclusion $\mathfrak{M} \models \mathbf{G}\varphi$. $\square$

Note that it is not valid worldwise! worldwise validity would mean that $\varphi \to \Box\varphi$ is valid. But that is not true.

Derivations in PC

AXIOM SYSTEM PC – FREGE-HILBERT STYLE

Three axiom **schemes**, one rule (called *modus ponens*)

$$
\begin{array}{ll}
\text{(A1)} & \varphi \to (\psi \to \varphi) \overset{\text{def}}{=} \{\varphi \to (\psi \to \varphi) \;:\; \varphi, \psi \in \mathcal{L}\} \\
\text{(A2)} & (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \\
\text{(A3)} & (\neg\varphi \to \neg\psi) \to (\psi \to \varphi)
\end{array}
$$

$$
\text{(MP)} \quad \frac{\begin{array}{l}\varphi \\ \varphi \to \psi\end{array}}{\psi}
$$

Axiom system PC – Frege-Hilbert style

→ STANDARD NOTATIONAL CONVENTIONS

Three axiom **schemes**:

$$
\begin{array}{ll}
(A1) & \varphi \to \psi \to \varphi \\
(A2) & (\varphi \to \psi \to \chi) \to (\varphi \to \psi) \to \varphi \to \chi \\
(A3) & (\neg\varphi \to \neg\psi) \to \psi \to \varphi
\end{array}
$$

One rule (called *modus ponens*): $(MP) \quad \dfrac{\varphi \qquad \varphi \to \psi}{\psi}$

DEFINITION:: $\Gamma \vdash \varphi$ iff there is a finite sequence of formulas that ends with φ s.t. every preceding formula ψ_i is

- is a premise, i.e., $\psi_i \in \Gamma$
- is an instance of an axiom scheme, i.e., $\psi_i \in (A1) \cup (A2) \cup (A3)$
- is a result of a *modus ponens*, i.e., there are two preceding formulas of form ψ_j and $\psi_j \to \psi_i$.

$$
\begin{aligned}
\Gamma \vdash \varphi \overset{\text{def}}{\Leftrightarrow} & (\exists n \in \omega)(\exists \psi : n+1 \to \mathcal{L})(\psi_n = \varphi \wedge \\
& (\forall i < n)(\psi_i \in \Gamma \vee \psi_i \in (A1) \cup (A2) \cup (A3) \vee \\
& (\exists j < i)(\exists k < i)(\psi_k = (\psi_j \to \psi_i))))
\end{aligned}
$$

BASICS

REMARK:: Note that Γ can be infinite, but the proof (the sequence of formulas) itself must be finite. Logics with infinitary rules are special (maybe non-compact) logics where rules can have infinitely many premises. But even those proofs are finite.

DEFINITION::

$$\vdash \varphi \overset{\text{def}}{\Leftrightarrow} \emptyset \vdash \varphi \qquad \varphi \vdash \psi \overset{\text{def}}{\Leftrightarrow} \{\varphi\} \vdash \psi$$

PROPOSITION: **monotonicity**: more premises preserve consequence

$$\frac{\Gamma \vdash \varphi}{\Delta \cup \Gamma \vdash \varphi}$$

REMARK:: Tautologies are consequences of everything.

$$\vdash \varphi \implies \forall \Gamma \; \Gamma \vdash \varphi$$

$\vdash \varphi \to \varphi$

$\vdash$	$\varphi \to (A \to \varphi) \to \varphi$	(A1)	$\psi := A \to \varphi$
$\vdash$	$(\varphi \to (A \to \varphi) \to \varphi) \to (\varphi \to A \to \varphi) \to \varphi \to \varphi$	(A2)	$\psi := A \to \varphi$, $\chi := \varphi$
$\vdash$	$(\varphi \to A \to \varphi) \to \varphi \to \varphi$	(MP)	
$\vdash$	$\varphi \to A \to \varphi$	(A1)	$\psi := A$
$\vdash$	$\varphi \to \varphi$	(MP)	

Deduction theorem

$$\frac{\Gamma \cup \{\varphi\} \quad \vdash \quad \psi}{\Gamma \quad \vdash \quad \varphi \to \psi}$$

Easy direction ($\Uparrow$) is basically the modus ponens itself:

$$\begin{array}{rcll} \Gamma & \vdash & \varphi \to \psi & \\ \Gamma \cup \{\varphi\} & \vdash & \varphi \to \psi & \text{monotonicity} \\ \Gamma \cup \{\varphi\} & \vdash & \psi & \text{(MP)} \end{array}$$

The other direction ($\Downarrow$) can be done by induction on the length of proofs!

Deduction theorem

proofs of length $n = 1$

$$\frac{\Gamma \cup \{\varphi\} \quad \vdash \quad \psi}{\Gamma \quad \vdash \quad \varphi \to \psi}$$

First we prove the formula for proofs of length 1. That means that ψ is either a premise from Γ, or the premise φ or it is an axiom.
If it is φ itself, we have to prove that $\Gamma \vdash \varphi \to \varphi$, but that comes via the monotonicity from the previous slide.
If ψ is an axiom or a premise from Γ, then it can be derived from Γ:

$$\begin{array}{lll} \Gamma & \vdash & \psi \\ \Gamma & \vdash & \psi \to \varphi \to \psi \quad \text{(A1)} \\ \Gamma & \vdash & \varphi \to \psi \quad \text{(MP)} \end{array}$$

DEDUCTION THEOREM

PROOFS OF LENGTH $n > 1$

If $\Gamma \cup \{\varphi\} \vdash \psi$ but there is no 1-length proof, then (by the definition of proofs) there must be two previous formulas for which

$$\Gamma \cup \{\varphi\} \vdash \chi \qquad \text{and} \qquad \Gamma \cup \{\varphi\} \vdash \chi \to \psi$$

But now the proof of these formulas are shorter than n, we can use the (strong) induction hypothesis to prove that

$$\Gamma \vdash \varphi \to \chi \qquad \text{and} \qquad \Gamma \vdash \varphi \to \chi \to \psi$$

$$\begin{array}{lll l}
\Gamma & \vdash & \varphi \to \chi \to \psi & \text{right} \\
\Gamma & \vdash & (\varphi \to \chi \to \psi) \to (\varphi \to \chi) \to \varphi \to \psi & \text{(A2)} \\
\Gamma & \vdash & (\varphi \to \chi) \to \varphi \to \psi & \text{(MP)} \\
\Gamma & \vdash & \varphi \to \chi & \text{left} \\
\Gamma & \vdash & \varphi \to \psi & \text{(MP)}
\end{array}$$

CUT (FREGE-HILBERT-STYLE)

$$\frac{\Gamma \vdash \varphi \qquad \Delta \cup \{\varphi\} \vdash \psi}{\Gamma \cup \Delta \vdash \psi}$$

$$\begin{array}{rcl} \Delta \cup \{\varphi\} & \vdash & \psi \\ \Delta & \vdash & \varphi \to \psi \\ \Gamma \cup \Delta & \vdash & \varphi \to \psi \\ \Gamma & \vdash & \varphi \\ \Gamma \cup \Delta & \vdash & \varphi \\ \Gamma \cup \Delta & \vdash & \psi \end{array}$$

Chain-rule

$$\frac{\begin{array}{ccc}\Gamma & \vdash & \varphi \to \psi \\ \Gamma & \vdash & \psi \to \chi\end{array}}{\begin{array}{ccc}\Gamma & \vdash & \varphi \to \chi\end{array}}$$

Hint: Cut and Deduction.

Reverse contraposition

$$\frac{\Gamma \quad \vdash \quad \neg\varphi \to \neg\psi}{\Gamma \quad \vdash \quad \psi \to \varphi}$$

Homework

Negation rules

$$\vdash \neg\neg\varphi \to \varphi \qquad \vdash \varphi \to \neg\neg\varphi$$

Hint:

$$\{\neg\neg\varphi, \neg\neg\neg\neg\varphi\} \vdash \neg\neg\varphi$$

and after that prove $\neg\neg\varphi \to \varphi$ first and use that to the other.

Contraposition

$$\frac{\Gamma \quad \vdash \quad \varphi \to \psi}{\Gamma \quad \vdash \quad \neg\psi \to \neg\varphi}$$

Hint: Negation rules + Chain-rule

EX FALSO QUODLIBET

$$\{\varphi, \neg\varphi\} \vdash \psi$$

$$\bot \vdash \psi$$

Hint: (A1) with negated formulas and use reverse contraposition (A3)

$$\frac{\begin{array}{ccc}\Gamma & \vdash & \varphi \\ \Gamma & \vdash & \neg\varphi\end{array}}{\begin{array}{ccc}\Gamma & \vdash & \bot\end{array}}$$

Hint: Check for definition of $\neg$.

CONSISTENCE

THEOREM: Enriching premises with consequences preserves consistence.

$$\frac{\begin{array}{rcc} \Gamma & \vdash & \varphi \\ \Gamma & \nvdash & \bot \end{array}}{\begin{array}{rcc} \Gamma \cup \{\varphi\} & \nvdash & \bot \end{array}}$$

Hint: prove by contradiction, use reverse contraposition

PROVE BY CONTRADICTION

$$\frac{\Gamma \vdash \varphi}{\Gamma \cup \{\neg\varphi\} \vdash \bot}$$

Hint: $\Downarrow$: ex falso quodlibet. $\Uparrow$ Alternation theorem
Alternation theorem is:

$$\vdash (\neg\varphi \to \varphi) \to \varphi$$

For that start with modus ponens:

$$\{\neg\varphi, \neg\varphi \to \varphi\} \vdash \neg\varphi$$