Ockhamist axioms

O : Ockhamist bundled trees

PC: Classical logic

(PC1) $\varphi \to .\psi \to \varphi$

(PC2) $\varphi \to (\psi \to \chi) \to .(\varphi \to \psi) \to .\varphi \to \chi$

(PC3) $\varphi \to \psi \to .\neg\psi \to \neg\varphi$

(MP) $\dfrac{\varphi,\ \varphi \to \psi}{\psi}$

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$\mathbf{K4.3_{F,P}}$: Temporal logic of linear frames

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(C) $\mathbf{PG}\varphi \to \varphi$, $\mathbf{FH}\varphi \to \varphi$

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(.3) $\mathbf{H}(\underline{\mathbf{H}}\varphi \to \psi) \vee \mathbf{H}(\underline{\mathbf{H}}\psi \to \varphi)$,
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Irreflexivity rule:

(IRR) $\dfrac{(p \wedge \mathbf{H}\neg p) \to \varphi}{\varphi}$ where p does not occur in φ

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Full-blooded Ockhamist axioms

(UPP) $\varphi \to \Box\varphi$ where $\mathbf{F}$ does not occur in φ.
unpreventability of past

(WDC) $\varphi \to \mathbf{G}\Box\mathbf{P}\Diamond\varphi$
weak diagram completion

(WDC+) $(p \wedge \mathbf{H}\neg p \wedge \Box\varphi) \to$
$\to \mathbf{G}\Box\mathbf{H}((p \wedge \mathbf{H}\neg p) \to \varphi)$

(MB) $\mathbf{G}\bot \to \Box\mathbf{G}\bot$ maximality of branches

IRR-RULE

LEMMA: The IRR rule is valid:

$$(IRR) \qquad \frac{(p \wedge \mathbf{H}\neg p) \rightarrow \varphi}{\varphi} \qquad \text{where } p \text{ does not occur in } \varphi$$

PROOF: Suppose that $(p \wedge \mathbf{H}\neg p) \rightarrow \varphi$ is valid on a Kamp-frame $\mathfrak{K}$, i.e., true in all worlds w.r.t. any Kamp-valuation. Now take an arbitrary but fixed world w and Kamp-valuation V. We will prove that $\mathfrak{K}, V, w \models^{K} \varphi$

$$\begin{array}{rcll}
\mathfrak{K}, V[p \mapsto \{v : w \equiv v\}], w & \models^{K} & (p \wedge \mathbf{H}\neg p) \rightarrow \varphi & \text{assumption} \\
\mathfrak{K}, V[p \mapsto \{v : w \equiv v\}], w & \models^{K} & p \wedge \mathbf{H}\neg p & \text{By } V[p \mapsto \{v : w \equiv v\}] \\
\mathfrak{K}, V[p \mapsto \{v : w \equiv v\}], w & \models^{K} & \varphi & \text{modus ponens} \\
\mathfrak{K}, V, w & \models^{K} & \varphi & p \text{ did not occur in } \varphi
\end{array}$$

■

Completeness Plan

1 We construct an irreflexive submodel $\mathfrak{M}_{\mathbf{O}}^{\mathrm{IRR}}$ of the canonical Kamp model $\mathfrak{M}_{\mathbf{O}}$. We will prove that we can use that model to prove a completeness proof. $W_{\mathbf{O}}^{\mathrm{IRR}}$ will be those maximally **O**-consistent worlds that are at the same time *IRR theories*, which we will define later, but in the meantime, the following properties will show why do we focus on that property:

- irr. If Γ is a maximally **O**-consistent IRR theory, then it is not true that $\Gamma <_{\mathbf{O}} \Gamma$.
- cl.irr. If Γ is a maximally **O**-consistent IRR theory, then there is no maximally **O**-consistent IRR theory Γ' s.t. $\Gamma <_{\mathbf{O}} \Gamma'$ and $\Gamma \equiv_{\mathbf{O}} \Gamma$.
- (IRRExt) If Γ is **O**-consistent and an infinite number of atomic sentences does not occur in Γ, then it can be extended into an **O**-consistent IRR theory Γ^+.
- (IRRLin) If Γ is **O**-consistent and IRR, then it can be extended into maximally **O**-consistent IRR theory Γ^+.
- (L^-) If Γ is IRR, then so is $\mathrm{L}^-(\Gamma)$ for any $\mathrm{L} \in \{\Box, \mathbf{F}, \mathbf{H}\}$.
- (FE) If Γ is IRR, then so is $\Gamma \cup \{\varphi\}$ for any $\varphi \in \mathcal{L}_{\mathbf{O}}$.
- (Ex) If Γ is an max. **O**-con. IRR theory s.t. $\Diamond\varphi \in \Gamma$, then there is a Γ' s.t. $\Gamma R_{\mathbf{O}} \Gamma'$ and $\varphi \in \Gamma'$, where $R_{\mathbf{O}} \in \{<_{\mathbf{O}}, \equiv_{\mathbf{O}}\}$.
- (Truth) Truth is membership in the IRR submodel of the canonical model.
- (CMT^-) The IRR submodel of the canonical model is almost a Kamp model – canonicity proofs for all property except the maximality of histories.

2 We transform $\mathfrak{M}_{\mathbf{O}}^{\mathrm{IRR}}$ into an $\mathrm{MB}(\mathfrak{M}_{\mathbf{O}}^{\mathrm{IRR}})$ in which the histories are maximal.

3 We prove that $\mathfrak{M}_{\mathbf{O}}^{\mathrm{IRR}}$ is a zigzag image of $\mathrm{MB}(\mathfrak{M}_{\mathbf{O}}^{\mathrm{IRR}})$.
We can conclude a weak completeness theorem for Kamp semantics

4 We construct a bundled tree model $\mathrm{BT}(\mathrm{MB}(\mathfrak{M}_{\mathbf{O}}^{\mathrm{IRR}}))$ which satisfy the same formulas as $\mathrm{MB}(\mathfrak{M}_{\mathbf{O}}^{\mathrm{IRR}})$.
We can conclude a weak completeness theorem for bundled tree semantics

5 Strong completeness can be gained by enriching the language with countable infinite new propositional variable and to reconstruct the procedure above with that enriched language.

IRR theories

IRR-THEORIES: IDEAS

Let us consider a loop as a sin.

Γ can prove its innocence easily iff

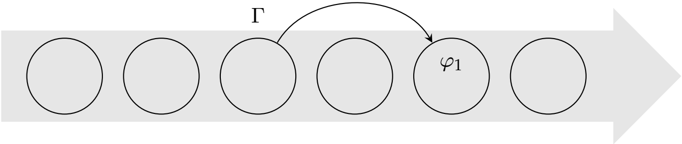
$$\Gamma \vdash_{\mathrm{O}} p \wedge \mathbf{H}\neg p \text{ for some } p \in \mathrm{At}.$$

Γ is in the company of easily provable innocents iff

$$\frac{\Gamma \vdash_{\mathrm{O}} \mathrm{M}_1(\varphi_1 \wedge \mathrm{M}_2(\varphi_2 \wedge \cdots \wedge \mathrm{M}_{n-1}(\varphi_{n-1} \wedge \mathrm{M}_n \varphi_n) \ldots))}{\begin{array}{l}\text{There is a } p \in \mathrm{At} \text{ not occurring in } \varphi_1, \ldots, \varphi_n \text{, s.t.} \\ \Gamma \vdash_{\mathrm{O}} \mathrm{M}_1(\varphi_1 \wedge \mathrm{M}_2(\varphi_2 \wedge \cdots \wedge \mathrm{M}_{n-1}(\varphi_{n-1} \wedge \mathrm{M}_n(\varphi_n \wedge p \wedge \mathbf{H}\neg p)) \ldots))\end{array}}$$

where $\mathrm{M}_i \in \{\Diamond, \mathbf{F}, \mathbf{P}\}$ for all $i \leq n$.

Consider $\varphi_1, \ldots, \varphi_n$ as tags of accessible worlds. The nested occurrences of "$\mathrm{M}_i(\varphi_i \wedge$" represents a search of the neighbour worlds where temporarily we tag every world with a formula that occurs there. The i-th step is made by M_i, and the tag of that world is φ_i.



We willl focus on those maximally **O**-consistent theories that can prove their innocence easily and are in the company of easily provable innocents. To do so, however, we will

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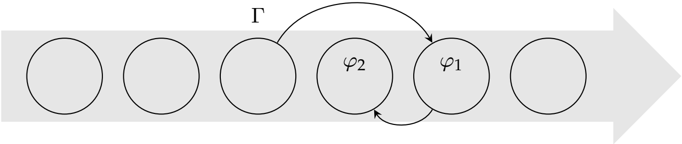

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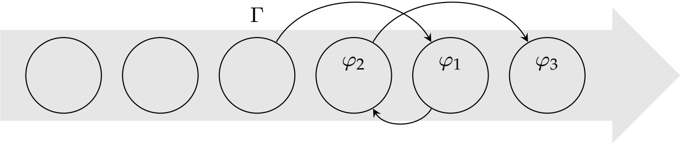

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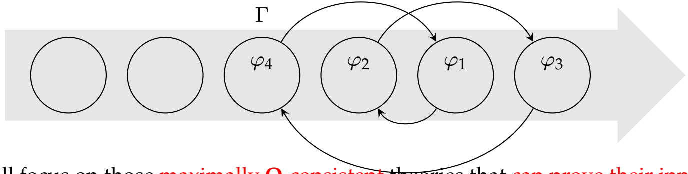

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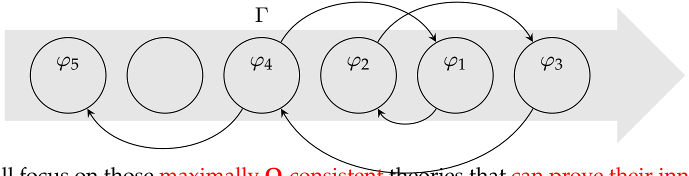

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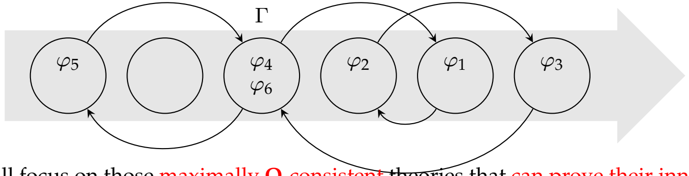

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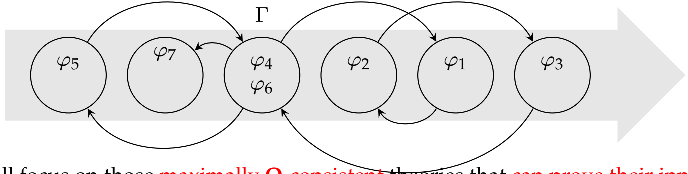

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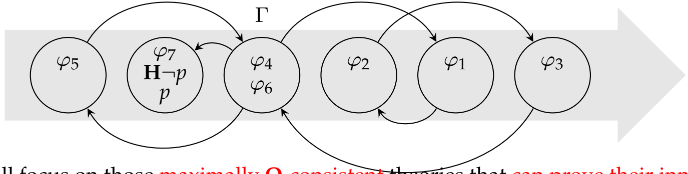

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IRR-THEORIES

Intuitively, Γ is an IRR-theory iff $\Gamma \vdash_O p \wedge \mathbf{H}\neg p$ for some $p \in \mathrm{At}$, and

$$\frac{\text{For all } p \in \mathrm{At} \text{ not occurring in } \varphi_1, \ldots, \varphi_n, \quad \Gamma \vdash_O \mathrm{L}_1(\varphi_1 \to \mathrm{L}_2(\varphi_2 \to \ldots \to \mathrm{L}_{n-1}(\varphi_{n-1} \to \mathrm{L}_n(\varphi_n \to (\neg(p \wedge \mathbf{H}\neg p)))) \ldots))}{\Gamma \vdash_O \mathrm{L}_1(\varphi_1 \to \mathrm{L}_2(\varphi_2 \to \ldots \to \mathrm{L}_{n-1}(\varphi_{n-1} \to \mathrm{L}_n(\varphi_n \to \bot)) \ldots))}$$

where $\mathrm{L}_i \in \{\Box, \mathbf{G}, \mathbf{H}\}$ for all $i \leq n$.

Precisely,

$$[\varnothing; \varnothing](\#) \overset{\text{def}}{=} \#,$$
$$[\langle \mathrm{L}, \vec{\mathrm{L}}\rangle; \langle \varphi, \vec{\varphi}\rangle](\#) \overset{\text{def}}{=} \mathrm{L}(\varphi \to [\vec{\mathrm{L}}; \vec{\varphi}](\#)) \text{ where } \mathrm{L} \in \{\Box, \mathbf{G}, \mathbf{H}\}$$

and $[\vec{\mathrm{L}}; \vec{\varphi}](\varphi) \overset{\text{def}}{=} [\vec{\mathrm{L}}; \vec{\varphi}](\#/\varphi)$. Then Γ is IRR iff

$$\frac{\text{For all } p \in \mathrm{At} \text{ not occurring in } \vec{\varphi}, \quad \Gamma \vdash_O [\vec{\mathrm{L}}; \vec{\varphi}](\neg(p \wedge \mathbf{H}\neg p))}{\Gamma \vdash_O [\vec{\mathrm{L}}; \vec{\varphi}](\bot)}$$

where $\vec{\mathrm{L}}$ is an n-tuple over $\{\Box, \mathbf{G}, \mathbf{H}\}$.

(IRREXT)

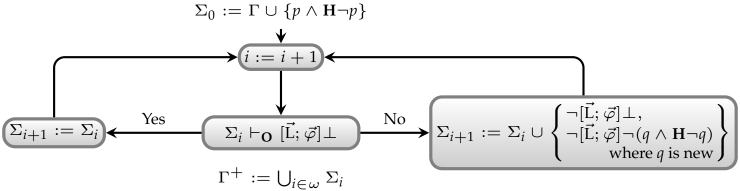
A consistent theory in which an infinite number of atomic propositions do not occur, can be extended to a consistent IRR theory.

PROOF:

So let Γ be a consistent theory described above, and let p an atom not occurring in Γ.

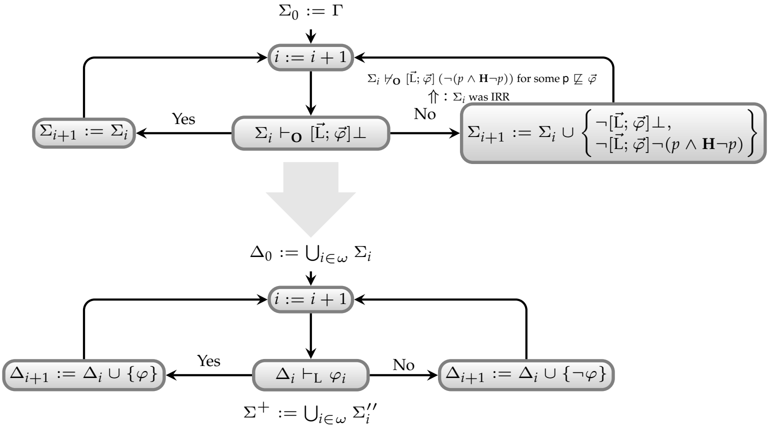
Let $\Sigma_0 \overset{\text{def}}{=} \Gamma \cup \{p \wedge \mathbf{H}\neg p\}$. This is consistent, for if

$\Gamma \cup \{p \wedge \mathbf{H}\neg p\}$	$\vdash$	$\bot$	ind.ass.
Γ	$\vdash$	$\neg(p \wedge \mathbf{H}\neg p)$	Ded.thm.
$\exists\Gamma_{\text{finite}} \supseteq \Gamma$	$\vdash$	$\neg(p \wedge \mathbf{H}\neg p)$	def.of $\vdash$
	$\vdash$	$\bigwedge \Gamma_{\text{finite}} \rightarrow \neg(p \wedge \mathbf{H}\neg p)$	def.of $\vdash$
	$\vdash$	$(p \wedge \mathbf{H}\neg p) \rightarrow \neg \bigwedge \Gamma_{\text{finite}}$	contraposition
	$\vdash$	$\neg \bigwedge \Gamma_{\text{finite}}$	IRR-rule
Γ_{finite}	$\vdash$	$\bot$	ded.thm
Γ	$\vdash$	$\bot$	QED



(IRRLin)

Every consistent IRR set is extendable to a maximally consistent IRR set.



(L^-)

For $\mathrm{L} \in \{\Box, \mathbf{G}, \mathbf{H}\}$,

$$\Gamma \text{ is IRR} \implies \mathrm{L}^-(\Gamma) \text{ is IRR}$$

<u>PROOF</u>: Let $\varphi \sqsubseteq \vec{\psi}$ denote that φ is a subformula of an element of $\vec{\psi}$:

$$
\begin{array}{rrll}
(\forall p \not\sqsubseteq \vec{\varphi}) & \mathrm{L}^-(\Gamma) & \vdash_O [\vec{\mathrm{L}};\vec{\varphi}](\neg(p \wedge \mathbf{H}\neg p)) & \text{assumption} \\
(\forall p \not\sqsubseteq \vec{\varphi}) & \mathrm{L}^-(\Gamma) & \vdash_O \top \to [\vec{\mathrm{L}};\vec{\varphi}](\neg(p \wedge \mathbf{H}\neg p)) & \text{PC} \\
(\forall p \not\sqsubseteq \vec{\varphi}) & \Gamma & \vdash_O \mathrm{L}(\top \to [\vec{\mathrm{L}};\vec{\varphi}](\neg(p \wedge \mathbf{H}\neg p))) & \mathrm{L}^-\text{-thing} \\
(\forall p \not\sqsubseteq \vec{\varphi}) & \Gamma & \vdash_O [\langle \mathrm{L}, \vec{\mathrm{L}}\rangle; \langle \top, \vec{\varphi}\rangle](\neg(p \wedge \mathbf{H}\neg p)) & \text{def.of templates} \\
 & \Gamma & \vdash_O [\langle \mathrm{L}, \vec{\mathrm{L}}\rangle; \langle \top, \vec{\varphi}\rangle](\bot) & \Gamma \text{ is IRR} \\
 & \Gamma & \vdash_O \mathrm{L}(\top \to [\vec{\mathrm{L}};\vec{\varphi}](\bot)) & \text{def.of templates} \\
 & \mathrm{L}^-(\Gamma) & \vdash_O \top \to [\vec{\mathrm{L}};\vec{\varphi}](\bot) & \mathrm{L}^-\text{-thing} \\
 & \mathrm{L}^-(\Gamma) & \vdash_O [\vec{\mathrm{L}};\vec{\varphi}](\bot) & \text{PC}
\end{array}
$$

(FE)

For $\mathrm{L} \in \{\Box, \mathbf{G}, \mathbf{H}\}$,

$$\Gamma \text{ is IRR} \implies \Gamma \cup \{\varphi\} \text{ is IRR}$$

PROOF: Let $\varphi \sqsubseteq \vec{\psi}$ denote that φ is a subformula of an element of $\vec{\psi}$:

$(\forall p \not\sqsubseteq \vec{\varphi}) \quad \Gamma \cup \{\varphi\} \vdash_O [\vec{\mathrm{L}}; \vec{\varphi}](\neg(p \wedge \mathbf{H}\neg p))$ — assumption

$(\forall p \not\sqsubseteq \vec{\varphi}) \quad \Gamma \cup \{\varphi\} \vdash_O [\langle \mathrm{L}_1, \mathrm{L}_2, \ldots, \mathrm{L}_n \rangle; \langle \varphi_1, \varphi_2, \ldots, \varphi_n \rangle](\neg(p \wedge \mathbf{H}\neg p))$ — focusing on the inside of $\vec{\varphi}$ and $\vec{\mathrm{L}}$

$(\forall p \not\sqsubseteq \vec{\varphi}) \quad \Gamma \cup \{\varphi\} \vdash_O \mathrm{L}_1(\varphi_1 \to [\langle \mathrm{L}_2, \ldots, \mathrm{L}_n \rangle; \langle \varphi_2, \ldots, \varphi_n \rangle](\neg(p \wedge \mathbf{H}\neg p)))$ — def.of templates

$(\forall p \not\sqsubseteq \vec{\varphi}) \quad \mathrm{L}_1^-(\Gamma \cup \{\varphi\}) \vdash_O \varphi_1 \to [\langle \mathrm{L}_2, \ldots, \mathrm{L}_n \rangle; \langle \varphi_2, \ldots, \varphi_n \rangle](\neg(p \wedge \mathbf{H}\neg p))$ — L_1-thing

$(\forall p \not\sqsubseteq \vec{\varphi}) \quad \mathrm{L}_1^-(\Gamma) \cup \mathrm{L}_1^-(\{\varphi\}) \vdash_O \varphi_1 \to [\langle \mathrm{L}_2, \ldots, \mathrm{L}_n \rangle; \langle \varphi_2, \ldots, \varphi_n \rangle](\neg(p \wedge \mathbf{H}\neg p))$ — L_1^- distributes over $\cup$.

(FE)

Now if φ is of form $\mathrm{L}_1\psi$, then

$$(\forall p \not\sqsubseteq \vec{\varphi}) \quad \mathrm{L}_1^-(\Gamma) \cup \mathrm{L}_1^-(\{\mathrm{L}_1\psi\}) \vdash_{\mathrm{O}} \varphi_1 \to [\langle \mathrm{L}_2, \ldots, \mathrm{L}_n\rangle; \langle \varphi_2, \ldots, \varphi_n\rangle](\neg(p \land \mathbf{H}\neg p))$$

L_1^- distributes over $\cup$.

$$(\forall p \not\sqsubseteq \vec{\varphi}) \quad \mathrm{L}_1^-(\Gamma) \cup \{\psi\} \vdash_{\mathrm{O}} \varphi_1 \to [\langle \mathrm{L}_2, \ldots, \mathrm{L}_n\rangle; \langle \varphi_2, \ldots, \varphi_n\rangle](\neg(p \land \mathbf{H}\neg p))$$

def.of L_1^-

$$(\forall p \not\sqsubseteq \vec{\varphi}) \quad \mathrm{L}_1^-(\Gamma) \vdash_{\mathrm{O}} (\psi \land \varphi_1) \to [\langle \mathrm{L}_2, \ldots, \mathrm{L}_n\rangle; \langle \varphi_2, \ldots, \varphi_n\rangle](\neg(p \land \mathbf{H}\neg p))$$

ded.thm

$$(\forall p \not\sqsubseteq \vec{\varphi}) \quad \Gamma \vdash_{\mathrm{O}} \mathrm{L}_1((\psi \land \varphi_1) \to [\langle \mathrm{L}_2, \ldots, \mathrm{L}_n\rangle; \langle \varphi_2, \ldots, \varphi_n\rangle](\neg(p \land \mathbf{H}\neg p)))$$

L_1-thing

$$(\forall p \not\sqsubseteq \vec{\varphi}) \quad \Gamma \vdash_{\mathrm{O}} [\langle \mathrm{L}_1, \mathrm{L}_2, \ldots, \mathrm{L}_n\rangle; \langle \psi \land \varphi_1, \varphi_2, \ldots, \varphi_n\rangle](\neg(p \land \mathbf{H}\neg p))$$

def.of templates

$$\Gamma \vdash_{\mathrm{O}} [\langle \mathrm{L}_1, \mathrm{L}_2, \ldots, \mathrm{L}_n\rangle; \langle \psi \land \varphi_1, \varphi_2, \ldots, \varphi_n\rangle](\bot)$$

Γ is IRR

$$\Gamma \vdash_{\mathrm{O}} \mathrm{L}_1((\psi \land \varphi_1) \to [\langle \mathrm{L}_2, \ldots, \mathrm{L}_n\rangle; \langle \varphi_2, \ldots, \varphi_n\rangle](\bot))$$

def.of templates

$$\mathrm{L}_1^-(\Gamma) \vdash_{\mathrm{O}} (\psi \land \varphi_1) \to [\langle \mathrm{L}_2, \ldots, \mathrm{L}_n\rangle; \langle \varphi_2, \ldots, \varphi_n\rangle](\bot)$$

L_1-thing

$$\mathrm{L}_1^-(\Gamma) \cup \{\psi\} \vdash_{\mathrm{O}} \varphi_1 \to [\langle \mathrm{L}_2, \ldots, \mathrm{L}_n\rangle; \langle \varphi_2, \ldots, \varphi_n\rangle](\bot)$$

ded.thm

$$\mathrm{L}_1^-(\Gamma) \cup \mathrm{L}_1^-(\{\mathrm{L}_1\psi\}) \vdash_{\mathrm{O}} \varphi_1 \to [\langle \mathrm{L}_2, \ldots, \mathrm{L}_n\rangle; \langle \varphi_2, \ldots, \varphi_n\rangle](\bot)$$

ded.thm

$$\mathrm{L}_1^-(\Gamma) \cup \mathrm{L}_1(\{\varphi\}) \vdash_{\mathrm{O}} \varphi_1 \to [\langle \mathrm{L}_2, \ldots, \mathrm{L}_n\rangle; \langle \varphi_2, \ldots, \varphi_n\rangle](\bot)$$

def.of L_1^-

$$\mathrm{L}_1^-(\Gamma \cup \{\varphi\}) \vdash_{\mathrm{O}} \varphi_1 \to [\langle \mathrm{L}_2, \ldots, \mathrm{L}_n\rangle; \langle \varphi_2, \ldots, \varphi_n\rangle](\bot)$$

L_1^- distributes over $\cup$

$$\Gamma \cup \{\varphi\} \vdash_{\mathrm{O}} \mathrm{L}_1(\varphi_1 \to [\langle \mathrm{L}_2, \ldots, \mathrm{L}_n\rangle; \langle \varphi_2, \ldots, \varphi_n\rangle](\bot))$$

L_1-thing

$$\Gamma \cup \{\psi\} \vdash_{\mathrm{O}} [\langle \mathrm{L}_1, \mathrm{L}_2, \ldots, \mathrm{L}_n\rangle; \langle \varphi_1, \varphi_2, \ldots, \varphi_n\rangle](\bot)$$

def.of template

$$\Gamma \cup \{\psi\} \vdash_{\mathrm{O}} [\langle \vec{\mathrm{L}}\rangle; \langle \vec{\varphi}\rangle](\bot)$$

vector-notation

(FE)

Now if φ is NOT of form $L_1\psi$, then

$(\forall p \not\sqsubseteq \vec{\varphi})\quad L_1^-(\Gamma) \cup L_1^-(\{\varphi\}) \vdash_O \varphi_1 \to [\langle L_2, \dots, L_n\rangle; \langle \varphi_2, \dots, \varphi_n\rangle](\neg(p \wedge \mathbf{H}\neg p))$

L_1^- distributes over $\cup$.

$(\forall p \not\sqsubseteq \vec{\varphi})\quad L_1^-(\Gamma) \cup \varnothing \vdash_O \varphi_1 \to [\langle L_2, \dots, L_n\rangle; \langle \varphi_2, \dots, \varphi_n\rangle](\neg(p \wedge \mathbf{H}\neg p))$

def.of L_1^-

$(\forall p \not\sqsubseteq \vec{\varphi})\quad L_1^-(\Gamma) \vdash_O \varphi_1 \to [\langle L_2, \dots, L_n\rangle; \langle \varphi_2, \dots, \varphi_n\rangle](\neg(p \wedge \mathbf{H}\neg p))$

$(\forall p \not\sqsubseteq \vec{\varphi})\quad \Gamma \vdash_O L_1(\varphi_1 \to [\langle L_2, \dots, L_n\rangle; \langle \varphi_2, \dots, \varphi_n\rangle](\neg(p \wedge \mathbf{H}\neg p)))$

L_1-thing

$(\forall p \not\sqsubseteq \vec{\varphi})\quad \Gamma \vdash_O [\langle L_1, L_2, \dots, L_n\rangle; \langle \varphi_1, \varphi_2, \dots, \varphi_n\rangle](\neg(p \wedge \mathbf{H}\neg p))$

def.of templates

$\Gamma \vdash_O [\langle L_1, L_2, \dots, L_n\rangle; \langle \varphi_1, \varphi_2, \dots, \varphi_n\rangle](\bot)$ — Γ is IRR

$\Gamma \vdash_O L_1(\varphi_1 \to [\langle L_2, \dots, L_n\rangle; \langle \varphi_2, \dots, \varphi_n\rangle](\bot))$

def.of templates

$L_1^-(\Gamma) \vdash_O \varphi_1 \to [\langle L_2, \dots, L_n\rangle; \langle \varphi_2, \dots, \varphi_n\rangle](\bot)$ — L_1-thing

$L_1^-(\Gamma) \cup \{\varnothing\} \vdash_O \varphi_1 \to [\langle L_2, \dots, L_n\rangle; \langle \varphi_2, \dots, \varphi_n\rangle](\bot)$ — ded.thm

$L_1^-(\Gamma) \cup L_1^-(\{\varphi\}) \vdash_O \varphi_1 \to [\langle L_2, \dots, L_n\rangle; \langle \varphi_2, \dots, \varphi_n\rangle](\bot)$ — def.of L_1^-

$L_1^-(\Gamma \cup \{\varphi\}) \vdash_O \varphi_1 \to [\langle L_2, \dots, L_n\rangle; \langle \varphi_2, \dots, \varphi_n\rangle](\bot)$ — L_1^- distributes...

$\Gamma \cup \{\varphi\} \vdash_O L_1(\varphi_1 \to [\langle L_2, \dots, L_n\rangle; \langle \varphi_2, \dots, \varphi_n\rangle](\bot))$ — L_1-thing

$\Gamma \cup \{\psi\} \vdash_O [\langle L_1, L_2, \dots, L_n\rangle; \langle \varphi_1, \varphi_2, \dots, \varphi_n\rangle](\bot)$ — def.of template

$\Gamma \cup \{\psi\} \vdash_O [\vec{L}\rangle; \langle\vec{\varphi}\rangle](\bot)$ — vector-notation

Irreflexive canonical submodel

Canonical Kamp model

$$\mathfrak{M}_{\mathbf{O}} \stackrel{\text{def}}{=} (W_{\mathbf{O}}, <_{\mathbf{O}}, \equiv_{\mathbf{O}}, V_{\mathbf{O}})$$

where

- $W_{\mathbf{O}} \stackrel{\text{def}}{=} \{\Gamma : \Gamma$ is a maximally **O**-consistent IRR-theory$\}$,
- $\Gamma <_{\mathbf{O}} \Gamma'$ iff $\mathbf{G}^{-}(\Gamma) \subseteq \Gamma'$ Remember that these are equivalent:

$$\begin{aligned} \mathbf{G}^{-}(\Gamma) &\subseteq \Gamma' \\ \Gamma &\supseteq \mathbf{F}^{+}(\Gamma') \\ \Gamma &\supseteq \mathbf{H}^{-}(\Gamma') \\ \mathbf{P}^{+}(\Gamma) &\subseteq \Gamma' \end{aligned}$$

- $\Gamma \equiv_{\mathbf{O}} \Gamma'$ iff $\Box^{-}(\Gamma) \subseteq \Gamma'$, Similarly: $\begin{aligned} \Box^{-}(\Gamma) &\subseteq \Gamma' \\ \Gamma &\supseteq \Diamond^{+}(\Gamma') \end{aligned}$
- $\Gamma \in V_{\mathbf{O}}(p) \stackrel{\text{def}}{\Leftrightarrow} p \in \Gamma$.

(Ex)

Let M denote the dual pair of L.

$$M\varphi \in \Gamma \Longrightarrow (\exists \Gamma' \in W_{\mathbf{O}})\left[\Gamma' \supseteq L^{-}(\Gamma) \text{ and } \varphi \in \Gamma'\right]$$

Since Γ is IRR, so are $L^{-}(\Gamma)$ and $L^{-}(\Gamma) \cup \{\varphi\}$, by ($L^{-}$) and (FE). The latter is consistent by the standard argumentation:

$L^{-}(\Gamma) \cup \{\varphi\}$	$\vdash_{\mathbf{O}}$	$\bot$	indirect assumption
$L^{-}(\Gamma)$	$\vdash_{\mathbf{O}}$	$\neg\varphi$	Deduction theorem
$\exists \vec{\chi}$	$\vdash_{\mathbf{O}}$	$\neg\varphi$	def.of $L^{-}(\Gamma) \vdash_{\mathbf{O}}$
	$\vdash_{\mathbf{O}}$	$\bigwedge \vec{\chi} \to \neg\varphi$	def.of $\vdash_{\mathbf{O}}$
	$\vdash_{\mathbf{O}}$	$L \bigwedge \vec{\chi} \to L\neg\varphi$	Lemmon
	$\vdash_{\mathbf{O}}$	$\bigwedge L\vec{\chi} \to L\neg\varphi$	A-axiom
$\bigwedge L\vec{\chi}$	$\vdash_{\mathbf{O}}$	$L\neg\varphi$	def.of $\vdash_{\mathbf{O}}$
Γ	$\vdash_{\mathbf{O}}$	$L\neg\varphi$	$\chi \in L^{-}(\Gamma) \Leftrightarrow L\chi \in \Gamma$
Γ	$\vdash_{\mathbf{O}}$	$\neg\mathbf{F}\varphi$	Duality
$\Gamma \cup \{\mathbf{F}\varphi\}$	$\vdash_{\mathbf{O}}$	$\bot$	Deduction theorem
Γ	$\vdash_{\mathbf{O}}$	$\bot$	by the assumption $\mathbf{F}\varphi \in \Gamma$. QED

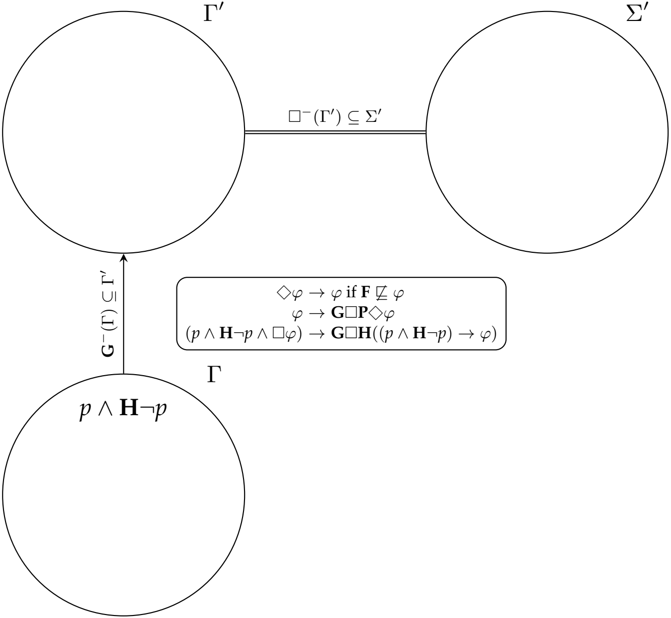
So we can extend $L^{-}(\Gamma) \cup \{\varphi\}$ by (IRRLin) into a maximally **O**-consistent IRR theory, and we are ready.

SUMMARY

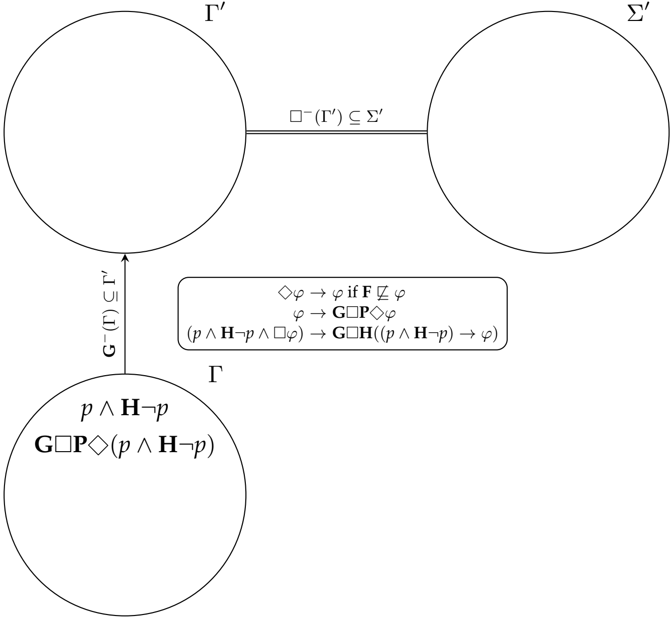
- The Truth Lemma goes through by our new existence lemma.
- $V_{\mathbf{O}}$ is a Kamp-valuation by the axiom of the unpreventability of past.

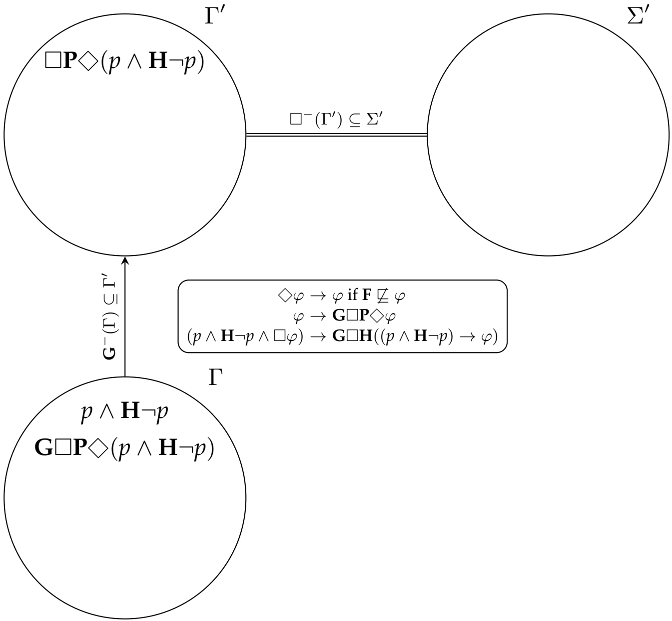
$$(UPP) \qquad \varphi \to \Box\varphi \text{ where } \mathbf{F} \text{ does not occur in } \varphi$$

- $<_{\mathbf{O}}$ is irreflexive by construction (all our canonical worlds are IRR theories).
- $<_{\mathbf{O}}$ is transitive and non-branching by the canonicity of 4 and .3.
- $\equiv_{\mathbf{O}}$ is reflexive, transitive and symmetric by the canonicity of T, 4 and B.
- $\Gamma \equiv_{\mathbf{O}} \Delta \to \Gamma \not<_{\mathbf{O}} \Delta$ comes from (UPP) and from the construction: There is a $p \wedge \mathbf{H}\neg p \in \Delta$, by UPP, $\Box(p \wedge \mathbf{H}\neg p) \in \Delta$, by def of $\equiv_{\mathbf{O}}$, and the symmetry of it, $p \wedge \mathbf{H}\neg p \in \Gamma$. But $\Gamma <_{\mathbf{O}} \Delta$ would mean that $\mathbf{H}^{-}(\Delta) \subseteq \Gamma$, so $\neg p \in \Gamma$ which causes a contradiction.
- $(w \equiv v \wedge w' < w) \to (\exists v' < v)\ w' \equiv v'$ – We prove this on the next slide
- $(\forall w, v)(\exists w' < w)(\exists v' < v)\ w \equiv v$ As in usual, we can take the generated submodels to validate this.
- $(\forall w, v)(w \equiv v \wedge w \neq v)(\exists w' > w)(\forall v' > v)\ w' \not\equiv v'$ that is not true, we will have to suffer with this later

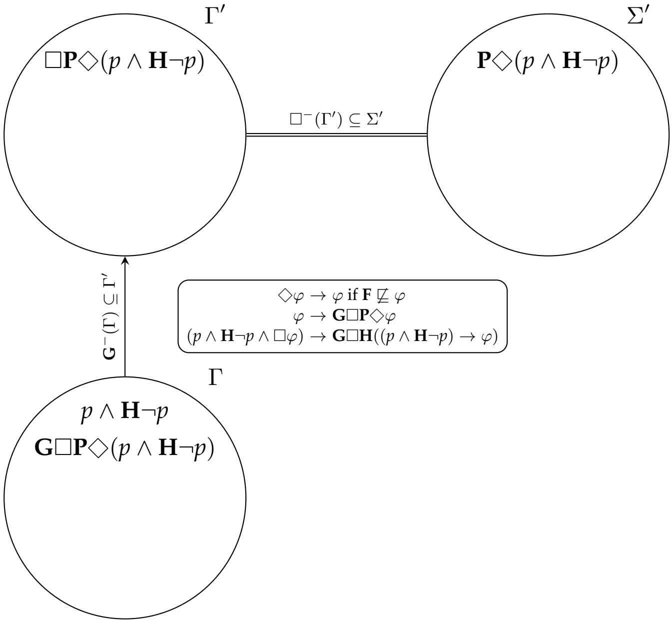


Γ′
Σ′
□⁻(Γ′) ⊆ Σ′
G⁻(Γ) ⊆ Γ′
◇φ → φ if F ⋢ φ
φ → G□P◇φ
(p ∧ H¬p ∧ □φ) → G□H((p ∧ H¬p) → φ)
Γ
p ∧ H¬p


Γ′
Σ′
□⁻(Γ′) ⊆ Σ′
G⁻(Γ) ⊆ Γ′
◇φ → φ if F ⊈ φ
φ → G□P◇φ
(p ∧ H¬p ∧ □φ) → G□H((p ∧ H¬p) → φ)
Γ
p ∧ H¬p
G□P◇(p ∧ H¬p)



Γ'
$\Box \mathbf{P} \Diamond (p \wedge \mathbf{H} \neg p)$
Σ'
$\Box^{-}(\Gamma') \subseteq \Sigma'$
$\mathbf{G}^{-}(\Gamma) \subseteq \Gamma'$
$\Diamond \varphi \to \varphi$ if $\mathbf{F} \not\sqsubseteq \varphi$
$\varphi \to \mathbf{G} \Box \mathbf{P} \Diamond \varphi$
$(p \wedge \mathbf{H} \neg p \wedge \Box \varphi) \to \mathbf{G} \Box \mathbf{H}((p \wedge \mathbf{H} \neg p) \to \varphi)$
Γ
$p \wedge \mathbf{H} \neg p$
$\mathbf{G} \Box \mathbf{P} \Diamond (p \wedge \mathbf{H} \neg p)$


Γ'
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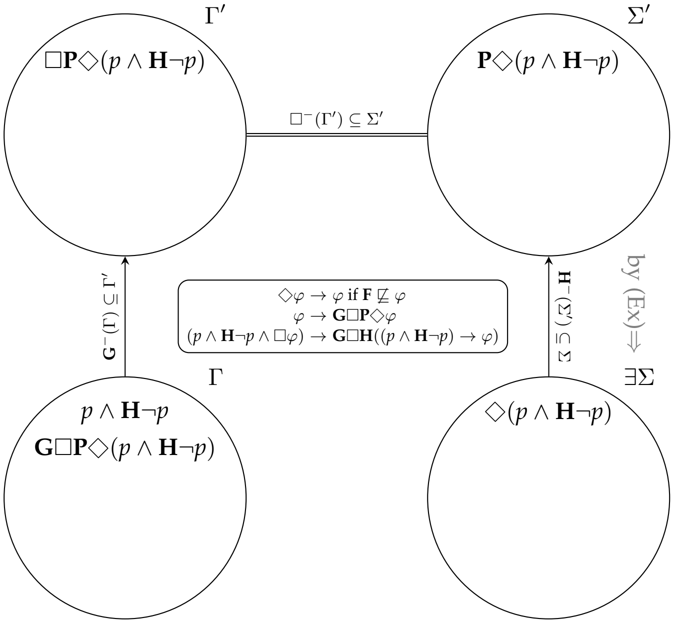
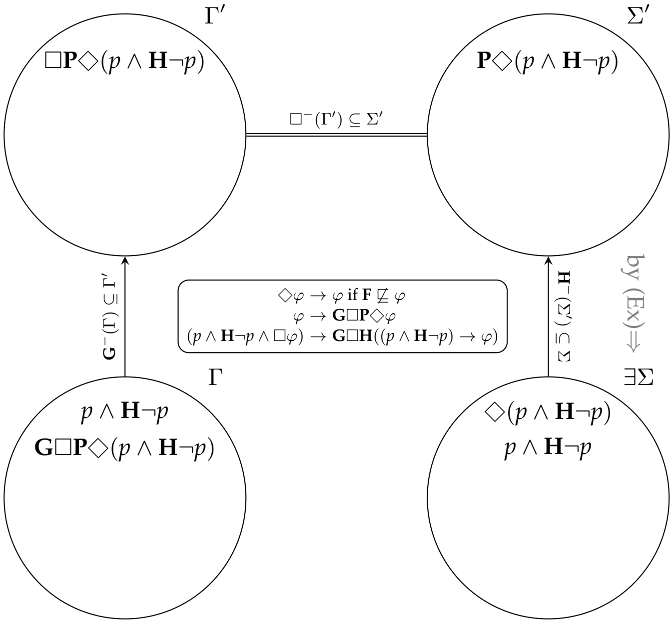
$(p \wedge \mathbf{H}\neg p \wedge \Box\varphi) \to \mathbf{G}\Box\mathbf{H}((p \wedge \mathbf{H}\neg p) \to \varphi)$
Γ
$p \wedge \mathbf{H}\neg p$
$\mathbf{G}\Box\mathbf{P}\Diamond(p \wedge \mathbf{H}\neg p)$


Γ′
□P◇(p ∧ H¬p)
Σ′
P◇(p ∧ H¬p)
□⁻(Γ′) ⊆ Σ′
G⁻(Γ) ⊆ Γ′
◇φ → φ if F ⊈ φ
φ → G□P◇φ
(p ∧ H¬p ∧ □φ) → G□H((p ∧ H¬p) → φ)
H⁻(Σ′) ⊆ Σ
by (Ex)⇒
Γ
p ∧ H¬p
G□P◇(p ∧ H¬p)
∃Σ
◇(p ∧ H¬p)


Γ'
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Σ'
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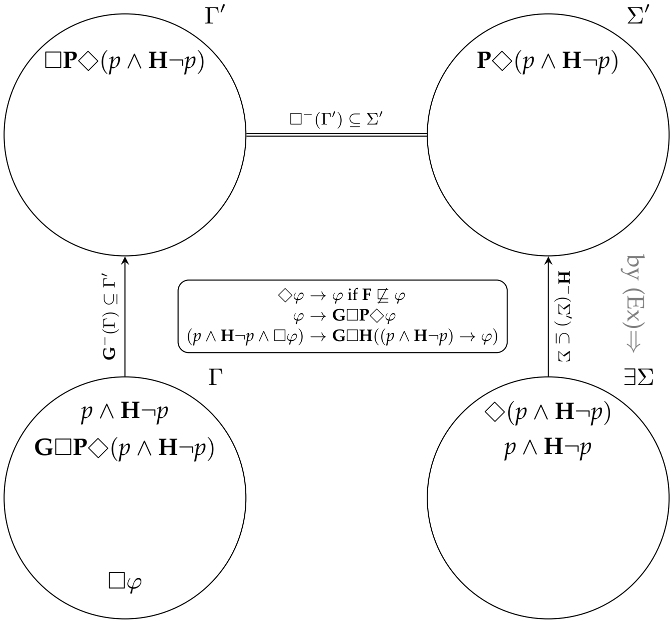
$(p \land \mathbf{H}\neg p \land \Box\varphi) \to \mathbf{G}\Box\mathbf{H}((p \land \mathbf{H}\neg p) \to \varphi)$
$\mathbf{H}^-(\Sigma') \supseteq \Sigma$
by (Ex)$\Rightarrow$
Γ
$p \land \mathbf{H}\neg p$
$\mathbf{G}\Box\mathbf{P}\Diamond(p \land \mathbf{H}\neg p)$
$\exists\Sigma$
$\Diamond(p \land \mathbf{H}\neg p)$
$p \land \mathbf{H}\neg p$


Γ'
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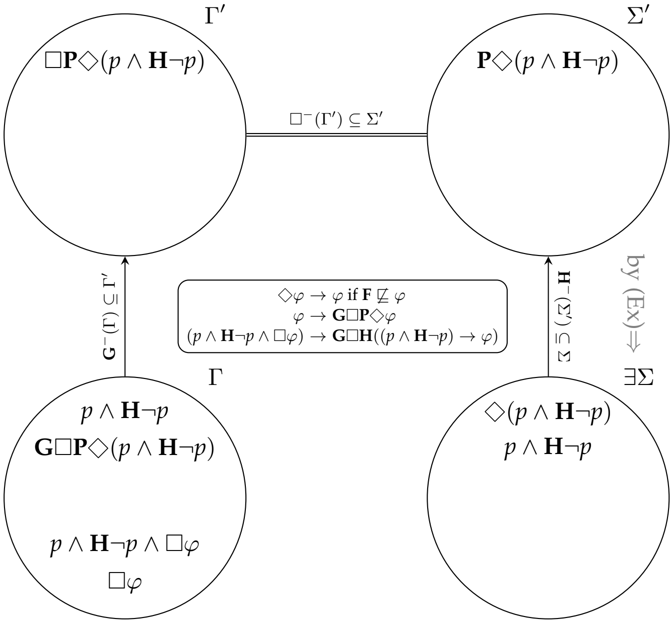
$(p \wedge \mathbf{H}\neg p \wedge \Box\varphi) \to \mathbf{G}\Box\mathbf{H}((p \wedge \mathbf{H}\neg p) \to \varphi)$
$\mathbf{H}^{-}(\Sigma') \subseteq \Sigma$
by (Ex)$\Rightarrow$
Γ
$\exists\Sigma$
$p \wedge \mathbf{H}\neg p$
$\mathbf{G}\Box\mathbf{P}\Diamond(p \wedge \mathbf{H}\neg p)$
$\Box\varphi$
$\Diamond(p \wedge \mathbf{H}\neg p)$
$p \wedge \mathbf{H}\neg p$


Γ'
$\Box\mathbf{P}\Diamond(p \wedge \mathbf{H}\neg p)$
Σ'
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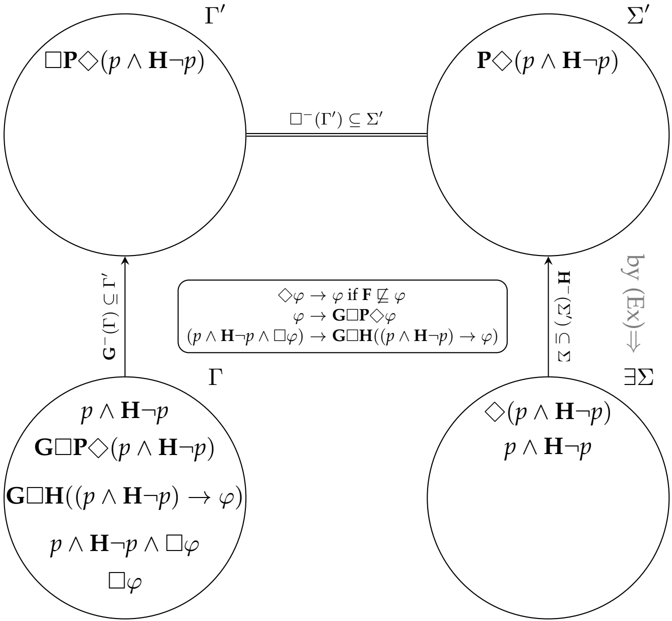
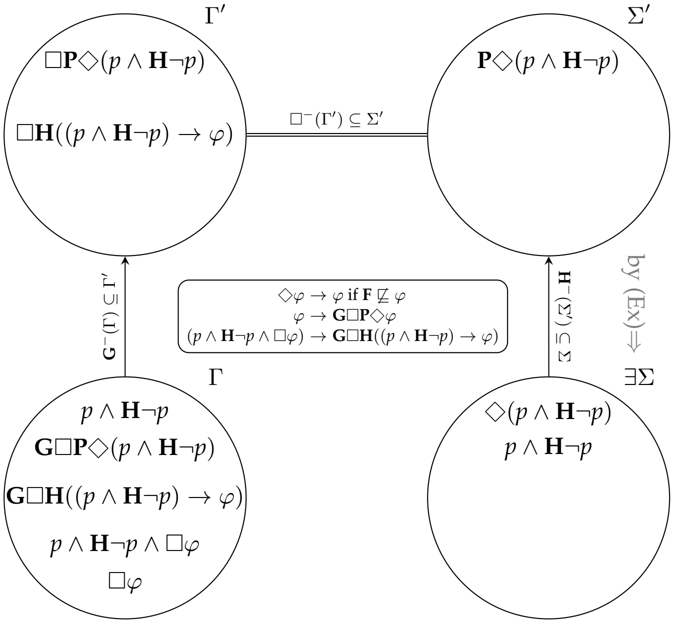
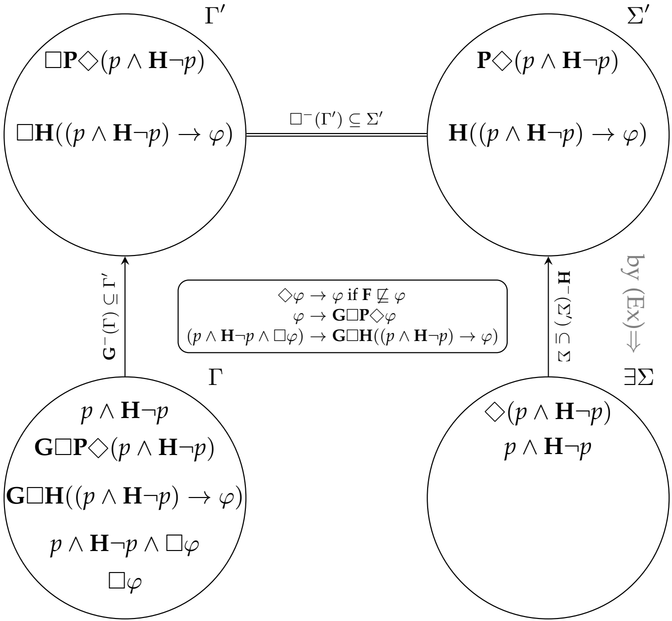
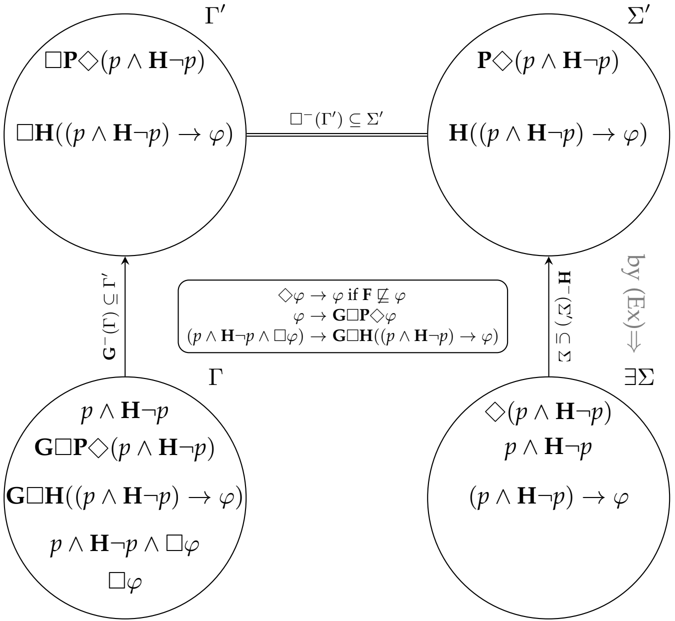
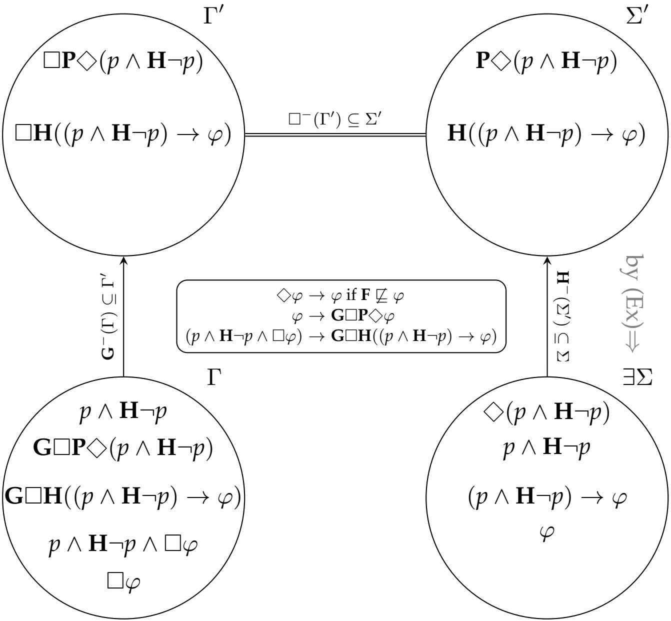
$\mathbf{H}^{-}(\Sigma') \subseteq \Sigma$
by (Ex) $\Rightarrow$
Γ
$\exists\Sigma$
$p \wedge \mathbf{H}\neg p$
$\mathbf{G}\Box\mathbf{P}\Diamond(p \wedge \mathbf{H}\neg p)$
$p \wedge \mathbf{H}\neg p \wedge \Box\varphi$
$\Box\varphi$
$\Diamond(p \wedge \mathbf{H}\neg p)$
$p \wedge \mathbf{H}\neg p$

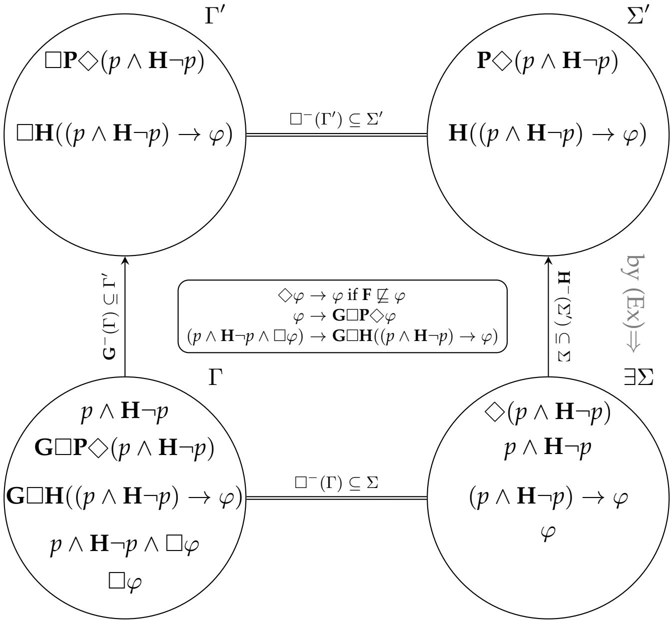

Γ'
$\Box\mathbf{P}\Diamond(p \wedge \mathbf{H}\neg p)$
Σ'
$\mathbf{P}\Diamond(p \wedge \mathbf{H}\neg p)$
$\Box^{-}(\Gamma') \subseteq \Sigma'$
$\mathbf{G}^{-}(\Gamma) \subseteq \Gamma'$
$\Diamond\varphi \rightarrow \varphi$ if $\mathbf{F} \not\sqsubseteq \varphi$
$\varphi \rightarrow \mathbf{G}\Box\mathbf{P}\Diamond\varphi$
$(p \wedge \mathbf{H}\neg p \wedge \Box\varphi) \rightarrow \mathbf{G}\Box\mathbf{H}((p \wedge \mathbf{H}\neg p) \rightarrow \varphi)$
$\mathbf{H}^{-}(\Sigma') \subseteq \Sigma$
by (Ex) $\Rightarrow$
Γ
$p \wedge \mathbf{H}\neg p$
$\mathbf{G}\Box\mathbf{P}\Diamond(p \wedge \mathbf{H}\neg p)$
$\mathbf{G}\Box\mathbf{H}((p \wedge \mathbf{H}\neg p) \rightarrow \varphi)$
$p \wedge \mathbf{H}\neg p \wedge \Box\varphi$
$\Box\varphi$
$\exists\Sigma$
$\Diamond(p \wedge \mathbf{H}\neg p)$
$p \wedge \mathbf{H}\neg p$


Γ'
$\Box\mathbf{P}\Diamond(p \wedge \mathbf{H}\neg p)$
$\Box\mathbf{H}((p \wedge \mathbf{H}\neg p) \to \varphi)$
Σ'
$\mathbf{P}\Diamond(p \wedge \mathbf{H}\neg p)$
$\Box^{-}(\Gamma') \subseteq \Sigma'$
$\mathbf{G}^{-}(\Gamma) \subseteq \Gamma'$
$\Diamond\varphi \to \varphi$ if $\mathbf{F} \not\sqsubseteq \varphi$
$\varphi \to \mathbf{G}\Box\mathbf{P}\Diamond\varphi$
$(p \wedge \mathbf{H}\neg p \wedge \Box\varphi) \to \mathbf{G}\Box\mathbf{H}((p \wedge \mathbf{H}\neg p) \to \varphi)$
$\mathbf{H}^{-}(\Sigma') \subseteq \Sigma$
by (Ex) $\Rightarrow$
Γ
$p \wedge \mathbf{H}\neg p$
$\mathbf{G}\Box\mathbf{P}\Diamond(p \wedge \mathbf{H}\neg p)$
$\mathbf{G}\Box\mathbf{H}((p \wedge \mathbf{H}\neg p) \to \varphi)$
$p \wedge \mathbf{H}\neg p \wedge \Box\varphi$
$\Box\varphi$
$\exists\Sigma$
$\Diamond(p \wedge \mathbf{H}\neg p)$
$p \wedge \mathbf{H}\neg p$


Γ′
□P◇(p ∧ H¬p)
□H((p ∧ H¬p) → φ)
□⁻(Γ′) ⊆ Σ′
Σ′
P◇(p ∧ H¬p)
H((p ∧ H¬p) → φ)
G⁻(Γ) ⊆ Γ′
◇φ → φ if F ⊈ φ
φ → G□P◇φ
(p ∧ H¬p ∧ □φ) → G□H((p ∧ H¬p) → φ)
H⁻(Σ′) ⊆ Σ
by (Ex) ⇒
Γ
p ∧ H¬p
G□P◇(p ∧ H¬p)
G□H((p ∧ H¬p) → φ)
p ∧ H¬p ∧ □φ
□φ
∃Σ
◇(p ∧ H¬p)
p ∧ H¬p


Γ'
$\Box\mathbf{P}\Diamond(p \wedge \mathbf{H}\neg p)$
$\Box\mathbf{H}((p \wedge \mathbf{H}\neg p) \to \varphi)$
Σ'
$\mathbf{P}\Diamond(p \wedge \mathbf{H}\neg p)$
$\mathbf{H}((p \wedge \mathbf{H}\neg p) \to \varphi)$
$\Box^{-}(\Gamma') \subseteq \Sigma'$
$\mathbf{G}^{-}(\Gamma) \subseteq \Gamma'$
$\Sigma \subseteq \mathbf{H}^{-}(\Sigma')$
by (Ex) $\Rightarrow$
$\Diamond\varphi \to \varphi$ if $\mathbf{F} \nsubseteq \varphi$
$\varphi \to \mathbf{G}\Box\mathbf{P}\Diamond\varphi$
$(p \wedge \mathbf{H}\neg p \wedge \Box\varphi) \to \mathbf{G}\Box\mathbf{H}((p \wedge \mathbf{H}\neg p) \to \varphi)$
Γ
$p \wedge \mathbf{H}\neg p$
$\mathbf{G}\Box\mathbf{P}\Diamond(p \wedge \mathbf{H}\neg p)$
$\mathbf{G}\Box\mathbf{H}((p \wedge \mathbf{H}\neg p) \to \varphi)$
$p \wedge \mathbf{H}\neg p \wedge \Box\varphi$
$\Box\varphi$
$\exists\Sigma$
$\Diamond(p \wedge \mathbf{H}\neg p)$
$p \wedge \mathbf{H}\neg p$
$(p \wedge \mathbf{H}\neg p) \to \varphi$


Γ'
$\Box\mathbf{P}\Diamond(p \wedge \mathbf{H}\neg p)$
$\Box\mathbf{H}((p \wedge \mathbf{H}\neg p) \to \varphi)$
Σ'
$\mathbf{P}\Diamond(p \wedge \mathbf{H}\neg p)$
$\mathbf{H}((p \wedge \mathbf{H}\neg p) \to \varphi)$
$\Box^{-}(\Gamma') \subseteq \Sigma'$
$\mathbf{G}^{-}(\Gamma) \subseteq \Gamma'$
$\Diamond\varphi \to \varphi$ if $\mathbf{F} \not\sqsubseteq \varphi$
$\varphi \to \mathbf{G}\Box\mathbf{P}\Diamond\varphi$
$(p \wedge \mathbf{H}\neg p \wedge \Box\varphi) \to \mathbf{G}\Box\mathbf{H}((p \wedge \mathbf{H}\neg p) \to \varphi)$
$\mathbf{H}^{-}(\Sigma') \subseteq \Sigma$
by (Ex) $\Rightarrow$
Γ
$\exists\Sigma$
$p \wedge \mathbf{H}\neg p$
$\mathbf{G}\Box\mathbf{P}\Diamond(p \wedge \mathbf{H}\neg p)$
$\mathbf{G}\Box\mathbf{H}((p \wedge \mathbf{H}\neg p) \to \varphi)$
$p \wedge \mathbf{H}\neg p \wedge \Box\varphi$
$\Box\varphi$
$\Diamond(p \wedge \mathbf{H}\neg p)$
$p \wedge \mathbf{H}\neg p$
$(p \wedge \mathbf{H}\neg p) \to \varphi$
φ


Γ′
□P◇(p ∧ H¬p)
□H((p ∧ H¬p) → φ)
Σ′
P◇(p ∧ H¬p)
H((p ∧ H¬p) → φ)
□⁻(Γ′) ⊆ Σ′
G⁻(Γ) ⊆ Γ′
H⁻(Σ′) ⊆ Σ
by (Ex) ⇒
◇φ → φ if F ⊈ φ
φ → G□P◇φ
(p ∧ H¬p ∧ □φ) → G□H((p ∧ H¬p) → φ)
Γ
p ∧ H¬p
G□P◇(p ∧ H¬p)
G□H((p ∧ H¬p) → φ)
p ∧ H¬p ∧ □φ
□φ
∃Σ
◇(p ∧ H¬p)
p ∧ H¬p
(p ∧ H¬p) → φ
φ
□⁻(Γ) ⊆ Σ

Maximalizing Histories